

# Spectrum and separability of mixed 2-qubit $X$ -states

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## Abstract

The separable mixed 2-qubit  $X$ -states are classified in accordance with degeneracies in the spectrum of density matrices. It is shown that there are four classes of separable  $X$ -states, among them: one 4D family, a pair of 2D family and a single, zero-dimensional maximally mixed state.

## Introduction

Consider the space  $\mathfrak{P}_X$  of  $4 \times 4$  Hermitian matrices of the form:

$$\varrho_X := \begin{pmatrix} \varrho_{11} & 0 & 0 & \varrho_{14} \\ 0 & \varrho_{22} & \varrho_{23} & 0 \\ 0 & \varrho_{32} & \varrho_{33} & 0 \\ \varrho_{41} & 0 & 0 & \varrho_{44} \end{pmatrix}. \quad (1)$$

Due to the Hermiticity, the diagonal entries in (1) are real numbers, while elements of the minor diagonal are pairwise complex conjugate numbers,  $\varrho_{14} = \overline{\varrho_{14}}$  and  $\varrho_{23} = \overline{\varrho_{32}}$ . Supposing that the matrix  $\varrho_X$  is semi-positive definite,

$$\varrho_X \geq 0, \quad (2)$$

and has a unit trace,

$$\text{tr} \varrho_X = 1, \quad (3)$$

the  $\varrho_X$  can be regarded as the density matrix of a 4-level quantum system. Since non-zero elements in (1) are distributed in a shape similar to the Latin letter “X”, the corresponding quantum states are named as  $X$ -states.

The 7-dimensional space  $\mathfrak{P}_X$  represents a subspace of the 15-dimensional state space  $\mathfrak{P}$  of a generic 4-level quantum system,  $\mathfrak{P}_X \subset \mathfrak{P}$ . Since the introduction of  $X$ -states [1], various subfamilies of  $\mathfrak{P}_X$  have been attracting a special attention. There are at least two reasons for that interest. First of all, it was found that microscopic systems, being in certain  $X$ -states, show a highly non-trivial quantum behaviour. <sup>1</sup> Secondly, due to the simple algebraic structure

<sup>1</sup>The well-known entangled states, such as Bell states [2], Werner states [3], isotropic states [4] and maximally entangled mixed states [5, 6], are particular subsets of  $X$ -states. For further references on  $X$ -states cf. [7], [8].

of  $X$ -states, many computational difficulties, common for generic states, can be resolved dealing with this special subclass of states.<sup>2</sup>

The aforementioned simplification turned out to be very important in describing such a complicated phenomenon as the entanglement in composite quantum systems. Particularly, it is well-known that the famous entanglement measure - concurrence - can be reduced to a simple analytical expression for  $X$ -states. In the present note we will move towards a detailed entanglement classification of the mixed 2-qubit  $X$ -states. Namely, the parametrization of separable mixed  $X$ -states of two qubits with an arbitrary spectrum of the density matrix will be described. Our analysis in the subsequent Sections includes the following steps:

1. Two unitary groups, both acting adjointly on the 7-dimensional space of 2-qubit  $X$ -states, will be introduced;
    - (a) The first one is the so-called “*global group*”,  $G_X \in SU(4)$ , defined as the invariance group of the subspace  $\mathfrak{P}_X$ ,
$$G_X \varrho_X G_X^\dagger \in \mathfrak{P}_X \quad \forall \quad \varrho_X \in \mathfrak{P}_X.$$
  - (b) The second one is the subgroup of  $G_X$ , the so-called “*local group*”,  $LG_X \in G_X$ . Its elements have a tensor product form corresponding to the decomposition of the state space  $\mathfrak{P}_X$  into two qubits subspaces,  $LG_X \in SU(2) \times SU(2)$ .
2. The “global orbits”,  $\mathcal{O}_\varrho$ , of the group  $G_X$  will be identified and classified into families/types according to the degeneracies in the spectrum of density matrices.
3. Considering the equivalence classes induced by the local group  $LG_X$  action on  $\mathcal{O}_\varrho$ , one can divide the latter into different subfamilies according to their entanglement characteristics. Having in mind this ranging, the separable density  $X$ -matrices will be categorized within the global orbits classification.

## 1 Global and local invariance groups of $X$ -states

In order to prove the properties of 2-qubit  $X$ -states announced above, let us start with few definitions.

• **Invariance subalgebra of  $X$ -states** • The basis for the  $\mathfrak{su}(4)$  algebra is constructed as follows: let  $\sigma_\mu = (\sigma_0, \boldsymbol{\sigma})$ s denote the set of  $2 \times 2$  matrices, where  $\sigma_0 = I$  is a unit matrix and  $\boldsymbol{\sigma} := (\sigma_x, \sigma_y, \sigma_z)$  are three Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The set of all possible tensor products of two copies of matrices  $\sigma_\mu$ ,

$$\sigma_{\mu\nu} := \sigma_\mu \otimes \sigma_\nu, \quad \mu, \nu = 0, x, y, z,$$

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<sup>2</sup>Such simplifications take place owing to a discrete symmetry  $X$ -states possess. Namely, it can be easily verified that every  $X$ -state (1) is equivalent to a block-diagonal matrix

$$\varrho_X = P_\pi \begin{pmatrix} \varrho_{11} & \varrho_{14} & 0 & 0 \\ \varrho_{41} & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{33} & \varrho_{32} \\ 0 & 0 & \varrho_{23} & \varrho_{22} \end{pmatrix} P_\pi, \quad \text{with} \quad P_\pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (4)$$

forms the basis of the algebra  $\mathfrak{su}(4)$ . For our aims it is useful to write the latter as the direct sum,  $\mathfrak{su}(4) = \mathfrak{l} \oplus \mathfrak{p}$ , where the 6-dimensional vector space  $\mathfrak{l}$  is composed as

$$\mathfrak{l} = \text{span} \frac{i}{2} \{ \sigma_{x0}, \sigma_{y0}, \sigma_{z0}, \sigma_{0x}, \sigma_{0y}, \sigma_{0z} \}, \quad (5)$$

while the 9-dimensional space  $\mathfrak{p}$  is <sup>3</sup>

$$\mathfrak{p} = \text{span} \frac{i}{2} \{ \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy}, \sigma_{zz} \}. \quad (6)$$

From now, to denote the matrices in (5) and (6), the notations  $\lambda_k$ , where  $k$  runs from 1 to 15, will be used

$$\mathfrak{l} = \text{span} \{ \lambda_1, \lambda_2, \dots, \lambda_6 \}, \quad \mathfrak{p} = \text{span} \{ \lambda_7, \lambda_8, \dots, \lambda_{15} \}. \quad (7)$$

$X$ -states (1) expand over the subset  $\alpha_X = \{ \lambda_{15}, \lambda_{10}, \lambda_6, -\lambda_{11}, \lambda_8, \lambda_3, \lambda_7 \}$  of the introduced  $\mathfrak{su}(4)$  basis:

$$\varrho_X = \frac{1}{4} \left( I + 2i \sum_{\lambda_k \in \alpha_X} h_k \lambda_k \right). \quad (8)$$

The real coefficients  $h_k$  in (8) are given by the linear combinations of the density matrix elements:

$$h_3 = -\varrho_{11} - \varrho_{22} + \varrho_{33} + \varrho_{44}, \quad h_6 = -\varrho_{11} + \varrho_{22} - \varrho_{33} + \varrho_{44}, \quad (9)$$

$$h_7 = -\varrho_{14} - \varrho_{23} - \varrho_{32} - \varrho_{41}, \quad h_{11} = -\varrho_{14} + \varrho_{23} + \varrho_{32} - \varrho_{41}, \quad (10)$$

$$h_8 = i(-\varrho_{14} + \varrho_{23} - \varrho_{32} + \varrho_{41}), \quad h_{10} = i(-\varrho_{14} - \varrho_{23} + \varrho_{32} + \varrho_{41}), \quad (11)$$

$$h_{15} = -\varrho_{11} + \varrho_{22} + \varrho_{33} - \varrho_{44}. \quad (12)$$

The subset  $\alpha_X$  possesses the following properties:

- i. The subset is closed under the matrix commutator operation, i.e., its elements span the subalgebra of  $\mathfrak{su}(4)$ ;
- ii. From the commutators collected in the Table 1. it follows that the element  $\lambda_{15}$  commutes with all other elements of  $\alpha_X$ ;
- iii. The remaining six elements,  $\{ \lambda_3, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, \lambda_{11} \}$ , span the  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ .

To check the last property, one can construct the following linear combinations:

$$S_z = i(\lambda_3 + \lambda_6), \quad S_{\pm} = \pm(\lambda_8 + \lambda_{10}) + i(\lambda_7 - \lambda_{11}), \quad (13)$$

$$T_z = i(\lambda_3 - \lambda_6), \quad T_{\pm} = \mp(\lambda_8 - \lambda_{10}) + i(\lambda_7 + \lambda_{11}), \quad (14)$$

and verify that their commutator relations read

$$[S_z, S_{\pm}] = \pm 2S_{\pm}, \quad [S_+, S_-] = 4S_z, \quad (15)$$

$$[T_z, T_{\pm}] = \pm 2T_{\pm}, \quad [T_+, T_-] = 4T_z. \quad (16)$$

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<sup>3</sup>Since the commutators between elements of two subspaces  $\mathfrak{l}$  and  $\mathfrak{p}$  are such that

$$[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}, \quad [\mathfrak{p}, \mathfrak{l}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l},$$

the direct sum  $\mathfrak{l} \oplus \mathfrak{p}$  is nothing else than the Cartan decomposition of  $\mathfrak{su}(4)$ .

Thus, two sets of elements

$$\mathbf{S} = \left\{ \frac{1}{2}(S_+ + S_-), \frac{i}{2}(S_+ - S_-), S_z \right\}, \quad (17)$$

$$\mathbf{T} = \left\{ \frac{1}{2}(T_+ + T_-), \frac{i}{2}(T_+ - T_-), T_z \right\} \quad (18)$$

generate two copies of  $\mathfrak{su}(2)$  algebra. <sup>4</sup> Gathering all together, we conclude that the set  $\alpha_X$  generates the subalgebra  $\mathfrak{g}_X := \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \in \mathfrak{su}(4)$ . <sup>5</sup>

• **Global unitary group of  $X$ -states** • Exponentiation of the algebra  $\mathfrak{g}_X$  results in the 7-parametric subgroup of  $SU(4)$ ,

$$G_X := \exp(\mathfrak{g}_X) \in SU(4),$$

whose action preserves the  $X$ -states space  $\mathfrak{P}_X$ , i.e.,  $G_X \varrho_X G_X^\dagger \in \mathfrak{P}_X$ . Using the expansion  $\mathfrak{g}_X = \sum_i \omega_i \lambda_i$  over the 7-tuple  $\lambda_i \in \alpha_X$  and the formulae (43)-(46) from the Section 5. **Supplementary material**, one can verify that the group  $G_X$  has the following representation:

$$G_X = P_\pi \left( \begin{array}{c|c} e^{-i\omega_{15}} SU(2) & 0 \\ \hline 0 & e^{i\omega_{15}} SU(2)' \end{array} \right) P_\pi, \quad (19)$$

where the two copies of  $SU(2)$  are parametrized as follows:

$$\begin{aligned} SU(2) &= \exp[i(\omega_4 + \omega_7)\sigma_1 + i(\omega_2 + \omega_5)\sigma_2 + i(\omega_3 + \omega_6)\sigma_3], \\ SU(2)' &= \exp[i(-\omega_4 + \omega_7)\sigma_1 + i(-\omega_2 + \omega_5)\sigma_2 + i(\omega_3 - \omega_6)\sigma_3]. \end{aligned}$$

• **Local subgroup of  $G_X$**  • Suppose now that our 4-level system is composed of 2-level subsystems, i.e., two qubits. In this case the Hilbert space  $\mathcal{H}$  is given by the tensor product of 2-dimensional Hilbert spaces,  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ , and one can consider the tensor product of operators acting independently on the subspaces of individual qubits,  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . Particularly, having in mind an intuitive idea of mutual independence of isolated qubits, we define the “*local unitary group*”,  $LG_X$ , as the subgroup of global invariance group of  $X$ -states,  $G_X$ , such that each of its elements  $g \in LG_X$  has the tensor product form,  $g = g_1 \times g_2$ , with  $g_1, g_2 \in SU(2)$ . From the expression (19) it follows that the local unitary group can be written as:

$$LG_X = P_\pi \exp(i \frac{\varphi_1}{2} \sigma_3) \times \exp(i \frac{\varphi_2}{2} \sigma_3) P_\pi. \quad (20)$$

## 2 Global $G_X$ -orbits

Now it will be shown that every  $X$ -states density matrix can be diagonalized using some subgroup of global  $G_X$  group. Therefore, the adjoint  $G_X$ -orbits structure is completely determined by the coset  $G_X/H_\varrho$ , where  $H_\varrho$  stands for the isotropy group of a density matrix  $\varrho$ . This isotropy group, in turn, depends on the degeneracies occurring in the spectrum of density matrices. Thus, the latter determines all possible types of  $G_X$ -orbits and the corresponding classification can be carried as follows.

<sup>4</sup>In terminology of [9] such operators describe “pseudospins” for two-spin system.

<sup>5</sup>For further information on a diverse algebraic structure of  $X$ -states see [10].

## 2.1 Dimensionality of the tangent space of $G_X$ -orbits

Consider the adjoint action of the global unitary group  $G_X$  on the 7-dimensional space  $\mathfrak{P}_X$  and introduce the following vectors at each point  $\varrho \in \mathfrak{P}_X$ :

$$t_k = \frac{\partial}{\partial v_k} (g(\mathbf{v}) \varrho_X g^\dagger(\mathbf{v})) \Big|_{v_k=0} = [\lambda_k, \varrho_X], \quad k = 3, 6, 7, 8, 10, 11, 15. \quad (21)$$

In the equation (21) the group elements  $g(\mathbf{v}) \in G_X$  are parametrized by 7-tuple  $\mathbf{v} = \{v_3, v_6, v_7, v_8, v_{10}, v_{11}, v_{15}\}$ :

$$g(\mathbf{v}) = \exp \left( \sum_{\lambda_k \in \alpha_X} v_k \lambda_k \right). \quad (22)$$

These vectors belong to a tangent space of  $G_X$ -orbits. The dimensionality of this tangent space is given by the rank of the  $7 \times 7$  Gram matrix

$$G = \|G_{kl}\| = \frac{1}{2} \|Tr(t_k t_l)\|. \quad (23)$$

Straightforward evaluation of the spectrum  $\sigma(G)$  of the Gram matrix  $G$  shows that it comprises: two eigenvalues of multiplicity 2 and three identically vanishing eigenvalues,

$$\sigma(G) = \{\mu_1, \mu_1, \mu_2, \mu_2, 0, 0, 0\}, \quad (24)$$

where the double multiplicity eigenvalues read:

$$\mu_1 = (h_3 + h_6)^2 + (h_8 + h_{10})^2 + (h_7 + h_{11})^2, \quad (25)$$

$$\mu_2 = (h_3 - h_6)^2 + (h_8 - h_{10})^2 + (h_7 - h_{11})^2. \quad (26)$$

The formulae (25) and (26) ensure that there exist 4 types of  $G_X$ -orbits:

- **dim  $\mathcal{O} = 4$** , the generic orbits;
- **dim  $\mathcal{O} = 2$** , the degenerate orbits defined by the equations:

$$h_6 = h_3, \quad h_{10} = h_8, \quad h_{11} = h_7; \quad (27)$$

- **dim  $\mathcal{O} = 2$** , the degenerate orbits defined by the equations:

$$h_6 = -h_3, \quad h_{10} = -h_8, \quad h_{11} = -h_7; \quad (28)$$

- **dim  $\mathcal{O} = 0$** , the single orbit  $\varrho_X = \frac{1}{4}I$  - the maximally mixed state.

In terms of the eigenvalues of density matrices, the 4D orbits are consistent with a generic spectrum, i.e., matrices with 4 different eigenvalues, while 2D orbits are generated by  $X$ -matrices with double multiplicity of the following types:

$$P_\pi \begin{pmatrix} \varrho_{11} & \varrho_{14} & 0 & 0 \\ \varrho_{41} & \varrho_{44} & 0 & 0 \\ 0 & 0 & \varrho_{22} & 0 \\ 0 & 0 & 0 & \varrho_{22} \end{pmatrix} P_\pi \text{ and } P_\pi \begin{pmatrix} \varrho_{11} & 0 & 0 & 0 \\ 0 & \varrho_{11} & 0 & 0 \\ 0 & 0 & \varrho_{33} & \varrho_{32} \\ 0 & 0 & \varrho_{23} & \varrho_{22} \end{pmatrix} P_\pi. \quad (29)$$

## 2.2 $G_X$ -orbits parametrization

Here a detailed representation for each type of  $G_X$ -orbits will be given, starting from the orbit of the highest dimensionality.

### 2.2.1 Generic orbits, $\dim(\mathcal{O}) = 4$

Let us assume that the spectrum of  $\varrho_X$  is a generic one, i.e., all eigenvalues  $\sigma(\varrho) := \{r_1, r_2, r_3, r_4\}$  are different positive real numbers. Furthermore, in the block-diagonal representation (4) of the density matrix  $\varrho_X$ , the  $\{r_1, r_2\}$  denote the eigenvalues of the upper block and  $\{r_3, r_4\}$  are eigenvalues of the lower block.

The  $4 \times 4$  density matrix  $\varrho_X$  can be diagonalized in a blockwise way,

$$\varrho_X = W \left( \begin{array}{c|c} \text{diag}(r_1, r_2) & 0 \\ \hline 0 & \text{diag}(r_3, r_4) \end{array} \right) W^\dagger, \quad (30)$$

using a special unitary matrix

$$W = P_\pi \left( \begin{array}{c|c} e^{i\omega} U & 0 \\ \hline 0 & e^{-i\omega} V \end{array} \right) P_\pi, \quad (31)$$

where  $U$  and  $V$  are  $2 \times 2$  special unitary matrices diagonalizing the upper and lower sub-blocks in (4). Since a generic spectrum has been assumed, matrices  $U$  and  $V$  belong to the coset,  $SU(2)/U(1) \times S_2$ , where the group  $S_2$  interchanges eigenvalues inside the pairs  $\{r_1, r_2\}$  and  $\{r_3, r_4\}$ . In order to have uniqueness in (30), one can fix a certain order in the spectrum  $\sigma(\varrho_X)$ . Namely, we assume that elements of the spectrum form a partially ordered simplex,  $\underline{\Delta}_3$ , i.e.,

$$\underline{\Delta}_3 : \sum_{i=1}^4 r_i = 1, \quad 0 \leq r_2 \leq r_1 \leq 1, \quad 0 \leq r_4 \leq r_3 \leq 1, \quad (32)$$

depicted in the FIGURE 1.<sup>6</sup>

Comparing expression (31) with (19), we convinced that the diagonalizing matrix is an element of the global group  $W \in G_X$  with  $2 \times 2$  special unitary matrices  $U$  and  $V$  from the coset  $SU(2)/U(1)$  parametrized by angles  $\phi_1, \phi_2 \in [0, \pi]$ ,  $\psi_1, \psi_2 \in [0, 2\pi]$ :

$$U = e^{i\frac{\psi_1}{2}\sigma_3} e^{i\frac{\phi_1}{2}\sigma_2}, \quad V = e^{i\frac{\psi_2}{2}\sigma_3} e^{i\frac{\phi_2}{2}\sigma_2}. \quad (33)$$

The 3-dimensional isotropy group  $H_{\text{Generic}}$  of generic orbits is

$$H_{\text{Generic}} = P_\pi \left( \begin{array}{c|c} e^{i\omega} \exp \frac{\gamma_1}{2} \sigma_3 & 0 \\ \hline 0 & e^{-i\omega} \exp \frac{\gamma_2}{2} \sigma_3 \end{array} \right) P_\pi. \quad (34)$$

This is in accordance with the maximal dimension of the  $G_X$ -orbits:

$$\dim(\mathcal{O})_{\text{Generic}} = \dim(G_X) - \dim H_{\text{Generic}} = 7 - 3 = 4.$$

Summarising, the adjoint action of the global group  $G_X$  determines the generic orbits, which are locally given by product of 2-spheres,  $S_2 \times S_2$ .

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<sup>6</sup>Note that the case of general position considered here consists of points inside the  $\underline{\Delta}_3$  and satisfies the inequalities  $r_2 < r_1$  and  $r_4 < r_3$ .

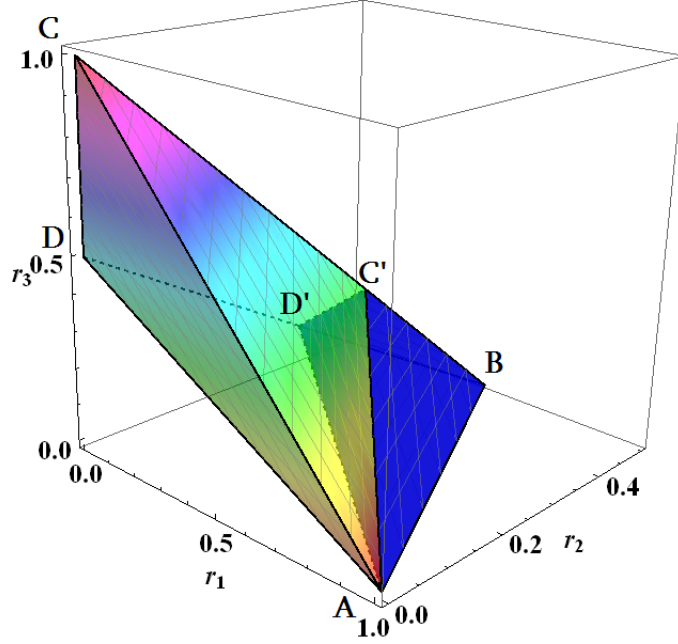


Figure 1: The tetrahedron  $ABCD$  as the image of the partially ordered simplex  $\underline{\Delta}_3$ , while the tetrahedron  $ABC'D'$  inside it corresponds to a 3D simplex with the following complete order of eigenvalues:  $\{ \sum_{i=1}^4 r_i = 1, \quad 1 \geq r_1 \geq r_2 \geq r_3 \geq r_4 \geq 0 \}$ .

### 2.2.2 Degenerate orbits, $\dim(\mathcal{O}) = 2$

According to the representation (29), two types of 2D degenerate  $G_X$ -orbits are generated by the matrices with degenerate  $2 \times 2$  sub-blocks, either upper or lower blocks. In the first case the isotropy group  $H_{\text{Degenerate}}$  reads

$$H_{\text{Degenerate}} = P_\pi \left( \begin{array}{c|c} e^{i\omega} SU(2) & 0 \\ \hline 0 & e^{-i\omega} \exp \frac{\gamma_2}{2} \sigma_3 \end{array} \right) P_\pi, \quad (35)$$

while for the second case  $H_{\text{Degenerate}}$  is

$$H'_{\text{Degenerate}} = P_\pi \left( \begin{array}{c|c} e^{i\omega} \exp \frac{\gamma_1}{2} \sigma_3 & 0 \\ \hline 0 & e^{-i\omega} SU(2)' \end{array} \right) P_\pi. \quad (36)$$

In both cases,  $\dim H_{\text{Degenerate}} = \dim H'_{\text{Degenerate}} = 5$  and the dimension of these degenerate  $G_X$ -orbits is

$$\dim(\mathcal{O})_{\text{Degenerate}} = \dim(G_X) - \dim H_{\text{Degenerate}} = 7 - 5 = 2.$$

### 2.2.3 Degenerate orbit, $\dim(\mathcal{O}) = 0$

Finally, there is one point in the state space  $\mathfrak{P}_X$ , whose isotropy group coincides with the invariance group  $G_X$ . This point corresponds to the maximally mixed state,  $\varrho_X = \frac{1}{4}I$ .

### 3 The separable states

Now we are in position to prove that every type of  $G_X$ -orbits includes the separable states.<sup>7</sup>

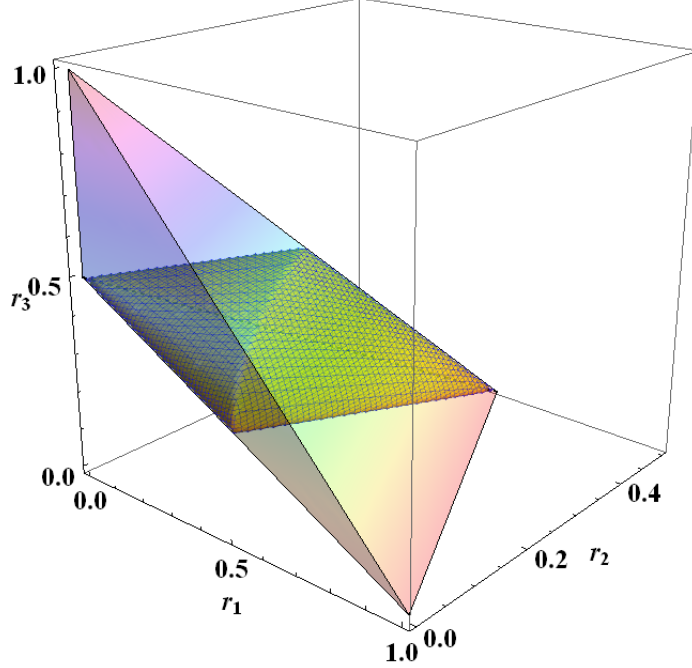


Figure 2: The absolute separable states inside the X-states tetrahedron.

#### 3.1 Separable states on the generic $G_X$ -orbits

The separability of states as a function of density matrices spectrum  $\sigma(\varrho_X)$ , can be analysed using the representation (30) for the generic  $G_X$ -orbits.

According to the Peres-Horodecki criterion [11], which is a necessary and sufficient condition for separability of  $2 \times 2$  and  $2 \times 3$  dimensional systems, a state  $\varrho$  is separable if its partial transposition, i.e.,  $\varrho^{T_2} = I \otimes T\varrho$ , is semi-positive as well.<sup>8</sup> Straightforward computation with  $\varrho_X$  in the form (30) shows that the semi-positivity of the partially transposed matrix  $\varrho_X^{T_2}$  requires

<sup>7</sup> The density matrix  $\varrho$  describing the mixed state of a composed system  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ , is *separable* if it allows the convex decomposition:

$$\varrho = \sum_k \omega_k \varrho_1^k \otimes \varrho_2^k, \quad \sum_k \omega_k = 1, \quad \omega_k > 0, \quad (37)$$

where  $\varrho_1^k$  and  $\varrho_2^k$  represent the density matrices acting on the multipliers  $\mathcal{H}_1$  and  $\mathcal{H}_2$  correspondingly. Otherwise, it is *entangled* [3].

<sup>8</sup> Here we consider the partial transposition with respect to the ordinary transposition operation  $T$  in the second subsystem; similarly, one can use the alternative action,  $\varrho^{T_1} = T \otimes I\varrho$ .



fulfilment of the following inequalities:

$$(r_1 - r_2)^2 \cos^2 \phi_1 + (r_3 - r_4)^2 \sin^2 \phi_2 \leq (r_1 + r_2)^2, \quad (38)$$

$$(r_3 - r_4)^2 \cos^2 \phi_2 + (r_1 - r_2)^2 \sin^2 \phi_1 \leq (r_3 + r_4)^2. \quad (39)$$

Note that the inequalities (38) and (39) do not constraint two angles  $\psi_1$  and  $\psi_2$  in (33) that parametrize the local group  $K = \exp(i\frac{\psi_1}{2}\sigma_3) \times \exp(i\frac{\psi_2}{2}\sigma_3)$ . It conforms with a general observation that the separability property is independent from the local characteristics of the composite system. This local group is the factor of the global group  $G_X = KG'_X$ , and the corresponding factor in the matrix  $W$  diagonalising  $\varrho_X$ , is irrelevant for the separability of  $X$ -states.

Analysing the inequalities (38) and (39), one can conclude:

- i. There are separable states for any values of eigenvalues from the partially ordered simplex  $\underline{\Delta}_3$ . In other words, the inequalities (38) and (39) determine non-empty domain of definition for angles  $\phi_1$  and  $\phi_2$  in (33) for every non-degenerate spectrum  $\sigma(\varrho_X)$ ;
- ii. There is a special family of the so-called “absolutely separable”  $X$ -states, such that the angles  $\phi_1$  and  $\phi_2$  can be arbitrary one. The absolutely separable  $X$ -states are generated by subset of the partially ordered simplex (32) defined by the inequalities

$$(r_1 - r_2)^2 \leq 4r_3r_4, \quad (40)$$

$$(r_3 - r_4)^2 \leq 4r_1r_2. \quad (41)$$

The FIGURE 2. illustrates location of the subset of the absolutely separable states inside the partially ordered simplex  $\underline{\Delta}_3$ .

### 3.2 Separable states on the degenerate $G_X$ -orbits

Testing the degenerate density matrices of the form (29) by the Peres-Horodecki criterion, we reveal the following picture. The positivity requirement of partially transposed density matrix with double multiplicity of eigenvalues gives the inequalities similar to (38) and (39). However, owing to the larger isotropy group  $H_{\text{Degenerate}}$  of states, the new inequalities depend solely on a single coordinate of the coset  $G_X/H_{\text{Degenerate}}$ . More precisely, if  $r_1 = r_2$ , i.e., the degeneracy occurs in the upper sub-block, then the angle  $\phi_2$  that parametrizes the matrix  $V$  in (33) plays the role of such a coordinate. In this case, the Peres-Horodecki criterion asserts that the degenerate  $X$ -state is separable iff:

$$\cos^2 \phi_2 \leq \frac{4\zeta}{(1 - \zeta)^2}, \quad (42)$$

where  $\zeta = r_4/r_3 < 1$ . This inequality points out the critical value  $\zeta_* = 3 - 2\sqrt{2}$ , such that for  $\zeta \leq \zeta_*$  the angle  $\phi_2$  is constrained, while for the interval  $\zeta_* < \zeta < 1$  the state is separable for an arbitrary angle  $\phi_2$ . The analogous results for the angle  $\phi_1$  (see the matrix  $U$  in (33)) hold true if the lower sub-block in (29) is degenerate, i.e.,  $r_3 = r_4$ . Therefore, in both classes of the degenerate 2D global orbits one can point out 2D family of separable degenerate states. Furthermore, among them there are the “degenerate absolutely separable” states, i.e., the degenerate global 2D orbits consisting completely from the separable states.

## 4 Concluding remarks

The present article is devoted to the discussion of an interplay between local and global characteristics of a pair of qubits in mixed  $X$ -states. With this aim, orbits of the global unitary group  $G_X$  action were described and classified according to the degeneracies occurring in the spectrum of density matrices. Based on this analysis, the dependence of  $X$ -states separability on the spectrum has been studied. Particularly, the separable  $X$ -states have been collected into the following families:

- The 4-dimensional family of separable states with the spectrum in general position;
- Two classes of 2-dimensional separable states with the double degeneracy spectrum;
- The maximally mixed state.

Finalizing notes, it is worth to comment that according to the aforementioned classification, the entangled states being complementary to the separable states, are partitioned likewise into three types. However, such classification is not complete. A further, more subtle ranging of the entangled states located on the given  $G_X$ -orbit into subclasses is necessary. The latter subclasses are determined not by invariants of the global group  $G_X$ , but are specified by the values of the  $LG_X$ -invariants. In the forthcoming publications we are planning to discuss this issue in more detail. Apart from that, following the approach elaborated in [12], [13] and [14], the generalization of the derived results for a generic case of 15-dimensional 2-qubit states will be considered.

## 5 Supplementary material

Here we collect a technical material useful for performing computations described in the main text. It includes the basis of the Lie algebra  $\mathfrak{su}(4)$ , commutators of its elements and block-diagonal representation for the subalgebra  $\alpha_X$ .

• **Basis for the Lie algebra  $\mathfrak{su}(4)$**  • The anti-Hermitian matrices,

$$\{\lambda_1, \lambda_2, \dots, \lambda_6\} = \frac{i}{2} \{\sigma_{x0}, \sigma_{y0}, \sigma_{z0}, \sigma_{0x}, \sigma_{0y}, \sigma_{0z}\}$$

and

$$\{\lambda_7, \lambda_8, \dots, \lambda_{15}\} = \frac{i}{2} \{\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yx}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{zy}, \sigma_{zz}\},$$

read:

$$\lambda_1 = \frac{i}{2} \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad \lambda_2 = \frac{i}{2} \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix}, \quad \lambda_3 = \frac{i}{2} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix},$$

$$\lambda_4 = \frac{i}{2} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \lambda_5 = \frac{i}{2} \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}, \quad \lambda_6 = \frac{i}{2} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix},$$

$$\lambda_7 = \frac{i}{2} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \lambda_8 = \frac{i}{2} \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \quad \lambda_9 = \frac{i}{2} \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix},$$

$$\lambda_{10} = \frac{i}{2} \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \quad \lambda_{11} = \frac{i}{2} \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad \lambda_{12} = \frac{i}{2} \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix},$$

$$\lambda_{13} = \frac{i}{2} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \quad \lambda_{14} = \frac{i}{2} \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}, \quad \lambda_{15} = \frac{i}{2} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

The block-diagonal form of basis elements of subalgebra  $\alpha_X$  resulting under transposition  $P_\pi$ :

$$P_\pi \lambda_3 P_\pi = \frac{i}{2} \left( \begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & -\sigma_3 \end{array} \right), \quad P_\pi \lambda_6 P_\pi = \frac{i}{2} \left( \begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & \sigma_3 \end{array} \right), \quad (43)$$

$$P_\pi \lambda_7 P_\pi = \frac{i}{2} \left( \begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & \sigma_1 \end{array} \right), \quad P_\pi \lambda_8 P_\pi = \frac{i}{2} \left( \begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & \sigma_2 \end{array} \right), \quad (44)$$

$$P_\pi \lambda_{10} P_\pi = \frac{i}{2} \left( \begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & -\sigma_2 \end{array} \right), \quad P_\pi \lambda_{11} P_\pi = \frac{i}{2} \left( \begin{array}{c|c} \sigma_1 & 0 \\ \hline 0 & -\sigma_1 \end{array} \right), \quad (45)$$

$$P_\pi \lambda_{15} P_\pi = \frac{i}{2} \left( \begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array} \right). \quad (46)$$

	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$	$\lambda_5$	$\lambda_6$	$\lambda_7$	$\lambda_8$	$\lambda_9$	$\lambda_{10}$	$\lambda_{11}$	$\lambda_{12}$	$\lambda_{13}$	$\lambda_{14}$	$\lambda_{15}$
$\lambda_1$	0	$-\lambda_3$	$\lambda_2$	0	0	0	0	0	0	$-\lambda_{13}$	$-\lambda_{14}$	$-\lambda_{15}$	$\lambda_{10}$	$\lambda_{11}$	$\lambda_{12}$
$\lambda_2$	$\lambda_3$	0	$-\lambda_1$	0	0	0	$\lambda_{13}$	$\lambda_{14}$	$\lambda_{15}$	0	0	0	$-\lambda_7$	$-\lambda_8$	$-\lambda_9$
$\lambda_3$	$-\lambda_2$	$\lambda_1$	0	0	0	0	$-\lambda_{10}$	$-\lambda_{11}$	$-\lambda_{12}$	$\lambda_7$	$\lambda_8$	$\lambda_9$	0	0	0
$\lambda_4$	0	0	0	0	$-\lambda_6$	$\lambda_5$	0	$-\lambda_9$	$\lambda_8$	0	$-\lambda_{12}$	$\lambda_{11}$	0	$-\lambda_{15}$	$\lambda_{14}$
$\lambda_5$	0	0	0	$\lambda_6$	0	$-\lambda_4$	$\lambda_9$	0	$-\lambda_7$	$\lambda_{12}$	0	$-\lambda_{10}$	$\lambda_{15}$	0	$\lambda_{13}$
$\lambda_6$	0	0	0	$-\lambda_5$	$\lambda_4$	0	$-\lambda_8$	$\lambda_7$	0	$-\lambda_{11}$	$\lambda_{10}$	0	$-\lambda_{14}$	$\lambda_{13}$	0
$\lambda_7$	0	$-\lambda_{13}$	$\lambda_{10}$	0	$-\lambda_9$	$\lambda_8$	0	$-\lambda_6$	$\lambda_5$	$-\lambda_3$	0	0	$\lambda_2$	0	0
$\lambda_8$	0	$-\lambda_{14}$	$\lambda_{11}$	$\lambda_9$	0	$-\lambda_7$	$\lambda_6$	0	$-\lambda_4$	0	$-\lambda_3$	0	0	$\lambda_2$	0
$\lambda_9$	0	$-\lambda_{15}$	$\lambda_{12}$	$-\lambda_8$	$\lambda_7$	0	$-\lambda_5$	$\lambda_4$	0	0	0	$-\lambda_3$	0	0	$\lambda_2$
$\lambda_{10}$	$\lambda_{13}$	0	$-\lambda_7$	0	$-\lambda_{12}$	$\lambda_{11}$	$\lambda_3$	0	0	0	$-\lambda_6$	$\lambda_5$	$-\lambda_1$	0	0
$\lambda_{11}$	$\lambda_{14}$	0	$-\lambda_8$	$\lambda_{12}$	0	$-\lambda_{10}$	0	$\lambda_3$	0	$\lambda_6$	0	$-\lambda_4$	0	$-\lambda_1$	0
$\lambda_{12}$	$\lambda_{15}$	0	$-\lambda_9$	$-\lambda_{11}$	$\lambda_{10}$	0	0	0	$\lambda_3$	$-\lambda_5$	$\lambda_4$	0	0	0	$-\lambda_1$
$\lambda_{13}$	$-\lambda_{10}$	$\lambda_7$	0	0	$-\lambda_{15}$	$\lambda_{14}$	$-\lambda_2$	0	0	$\lambda_1$	0	0	0	$-\lambda_6$	$\lambda_5$
$\lambda_{14}$	$-\lambda_{11}$	$\lambda_8$	0	$\lambda_{15}$	0	$-\lambda_{13}$	0	$-\lambda_2$	0	0	$\lambda_1$	0	$\lambda_6$	0	$-\lambda_4$
$\lambda_{15}$	$-\lambda_{12}$	$\lambda_9$	0	$-\lambda_{14}$	$\lambda_{13}$	0	0	0	$-\lambda_2$	0	0	$\lambda_1$	$-\lambda_5$	$\lambda_4$	0

Table 1: Commutator relations for  $\mathfrak{su}(4)$ .

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